

Problem Setup

Premise: To devise a sample-efficient linear-convergence algorithm for phase retrieval of (structured) s -sparse signals from Gaussian measurements.

Main Challenges

- ▶ Linear-convergence algorithms have sample complexity with quadratic dependence on sparsity $m = \mathcal{O}(s^2 \log n)$.
- ▶ High number of tuning parameters.
- ▶ Lower sample complexity algorithms require high run time, not scalable.

Prior Work

- ▶ *Convex:* PhaseLift, PhaseMax.
Drawbacks: Computationally expensive, poor empirical performance.
- ▶ *Non-convex:* AltMinPhase, Wirtinger Flow. Faster, scalable.
Drawbacks: Sample complexity depends on selecting good initial point. Parametric inputs.
- ▶ *Designed sensing matrices:* Matrices with low underlying dimension or Fourier-like.
Drawbacks: Harder to analyze theoretical guarantees.
- ▶ *Structured sparsity models:* Utilized for sparse signal recovery, statistical learning applications.
Drawbacks: No rigorous results for phase retrieval problem.

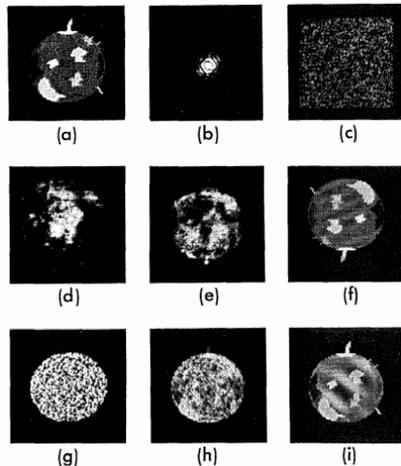


Fig. (a) Test object; (b) modulus of its Fourier transform; (c) initial estimate of the object (first test); (d)–(f) reconstruction results—number of iterations: (d) 20, (e) 230, (f) 600; (g) initial estimate of the object (second test); (h)–(i) reconstruction results—number of iterations: (h) 2, (i) 215.

Reproduced from [F78] J. Fienup, "Reconstruction of an object from the modulus of its Fourier transform." Optics letters, 1978.

Our Objective

We devise a phase-retrieval algorithm that:

- ▶ Utilizes underlying **structured sparsity** in signals for efficient analysis.
- ▶ Is **naturally compatible** with standard sparse recovery algorithms.
- ▶ Is **fast and scalable** to large datasets of large dimensions.
- ▶ Has **sub-quadratic** sample complexity $m = \mathcal{O}\left(\frac{s^2}{b} \log n\right)$.
- ▶ Requires **no extra parametric inputs** apart from (block) sparsity $k = \frac{s}{b}$.

New Direction: Phase retrieval of structured sparse signals

Idea: Efficient phase retrieval algorithms for structured sparse signals.

Problem Setup: Recover signal $\mathbf{x}^* \in \mathbb{R}^n$, using Gaussian sampling matrix $\mathbf{A} = [\mathbf{a}_1 \dots \mathbf{a}_m]^\top$, from measurements $\mathbf{y} \in \mathbb{R}^m$,

$$y_i = |\langle \mathbf{a}_i, \mathbf{x}^* \rangle|, \quad \text{for } i = 1, \dots, m.$$

\mathbf{x}^* is part of model $\mathcal{M}_{s,b}$ formed of uniformly block sparse signals with block length b , effective block sparsity $k = s/b$ and total number of blocks $n_b = n/b$.

Solution Methodology

Formulate the above as a two-step problem, by introducing diagonal phase matrix $\mathbf{P} \in \mathcal{P}$ with $P_{ij} = \text{sign}(\mathbf{a}_i^\top \mathbf{x}) \in \{1, -1\}$, and alternatively minimize the loss function over variables \mathbf{x} and \mathbf{P} :

$$\min_{\mathbf{x} \in \mathcal{M}, \mathbf{P} \in \mathcal{P}} \|\mathbf{A}\mathbf{x} - \mathbf{P}\mathbf{y}\|_2.$$

This strategy requires a good initial point, to converge to minimum. For this, we introduce our algorithm Block Compressive Phase Retrieval with Alternating Minimization (Block CoPRAM) [JH17].

Block CoPRAM - Smart Initialization

Setup: Define signal marginals as $M_{jj} = \frac{1}{m} \sum_{i=1}^m y_i^2 \mathbf{a}_{ij}^2$, for $j \in \{1 \dots n\}$.

Objective: Find good initial estimate \mathbf{x}^0 of the true signal \mathbf{x}^* .

Challenge: Designing block marginals. Performance guarantees.

Block CoPRAM - Smart Initialization (Cont.)

Solution: (Alg. 1)

- ▶ Define block marginals as $M_{j_b j_b} = \sqrt{\sum_{j \in j_b} M_{jj}^2}$, for $j_b \in \{1 \dots n_b\}$.
- ▶ Retain top k block marginals; call the index set \hat{S} , $\text{card}(\hat{S}) = s$.
- ▶ Construct matrix ($\in \mathbb{R}^{s \times s}$) $\mathbf{M}_{\hat{S}} = \frac{1}{m} \sum_{i=1}^m y_i^2 \mathbf{a}_{i\hat{S}} \mathbf{a}_{i\hat{S}}^\top$.
- ▶ Initial estimate $\mathbf{x}^0 = \phi \mathbf{v}$, \mathbf{v} is top-singular-vec($\mathbf{M}_{\hat{S}}$), where $\phi = \sqrt{\frac{1}{m} \sum_{i=1}^m y_i^2}$.

Guarantees: Theorem 1

The initial vector \mathbf{x}^0 , which is the output of Alg. 1, is a small constant distance δ_b away from the true signal \mathbf{x}^* , i.e.,

$$\text{dist}(\mathbf{x}^0, \mathbf{x}^*) \leq \delta_b \|\mathbf{x}^*\|_2,$$

where $0 < \delta_b < 1$, as long as the number of measurements satisfy $m \geq C \frac{s^2}{b} \log mn$ with probability greater than $1 - \frac{8}{m}$.

Block CoPRAM - Descent

Setup: Use the initialization \mathbf{x}^0 from Alg. 1.

Objective: Gradually descend to k -block sparse solution \mathbf{x}^* .

Challenge: Performance guarantees for convergence.

Solution: (Alg. 2)

Use alternating minimization with model CoSAMP to solve the non-convex problem:

- ▶ Phase estimation: $\mathbf{P}^t = \text{diag}(\text{sign}(\mathbf{A}\mathbf{x}^t))$.
- ▶ Signal estimation: $\mathbf{x}^t \approx \min_{\mathbf{x} \in \mathcal{M}_{s,b}} \|\mathbf{A}\mathbf{x} - \mathbf{P}^t \mathbf{y}\|_2$ via model CoSAMP.

Guarantees: Theorem 2

Given an initialization \mathbf{x}^0 satisfying Alg. 1, if number of measurements $m \geq C \left(s + \frac{s}{b} \log \frac{n}{s}\right)$, then the iterates of Alg. 2 satisfy:

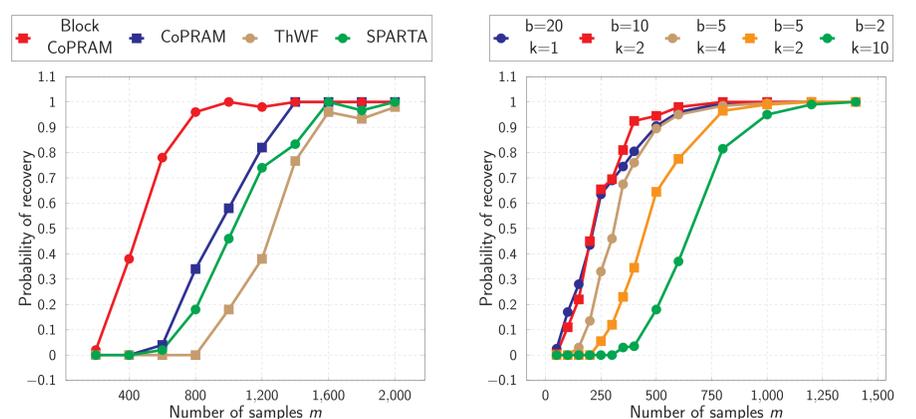
$$\text{dist}(\mathbf{x}^{t+1}, \mathbf{x}^*) \leq \rho_b \text{dist}(\mathbf{x}^t, \mathbf{x}^*).$$

where $0 < \rho_b < 1$ is a constant, with probability greater than $1 - e^{-\gamma m}$, for positive constant γ .

Results

Phase transitions: Block CoPRAM v/s CoPRAM, SPARTA, Thresholded Wirtinger Flow (ThWF).

CoPRAM is a special case of Block CoPRAM with $b = 1$.



(a) Sparsity $s = 30$ and block length $b = 5$. (b) Sparsity $s = 20$ and different block lengths (Block CoPRAM only).

Figure 1: Phase transitions for signal length $n = 3,000$.

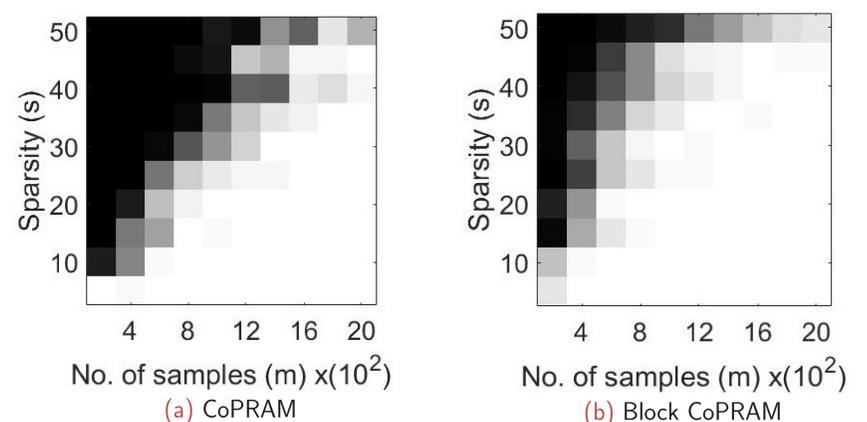


Figure 2: Phase transition for signal length $n = 3,000$ and block length $b = 5$ and different sparsity levels.

Acknowledgments

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